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Some Fixed Point Theorems in b_G-Metric Space

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Abstract:
A new metric which is derived from two metrics called G-metric and b-metric which were introduced by Mustafa and Sims [8] and Cezerwik [3], respectively. As an application of this newly developed space, we have established some fixed
Point theorems using this metric space.

I. INTRODUCTION AND PRELIMINARIES

The metric space has generalized by many ways. Czerwik [3] introduced the concept of b-metric space. We have noted various types of fixed point's results in [1, 5, 2, 4, 6]. Zead Mustafa and Brailey Sims [8] poised this and introduced the concept of G-metric space and proved some fixed point theorems. We have observed lot of fixed point in G-metric spaces in [7, 8, 9, 10, 11, 12, 13] from In this paper, we extend the G-metric space and develop the new structure of metric space, which we call b_G-metric space. Initially Cezerwik [3] defined b-metric space which follows:

Definition 1.1 Let X be a non empty set and the mapping $d: X \times X \to [0, \infty)$. The mapping d satisfies

(i) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) there exists a real numbers $s \geq 1$ such that $d(x, y) \leq s[d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a b-metric on X and the ordered pair (X, d) is called b-metric space with coefficients$s$.

For$s = 1$, the b-metric coincide with usual metric space. Z. Mustafa and B. Sim [8] introduced G-metric space as follows:

Definition 1.2 Let X be a non empty set and the mapping $G: X \times X \times X \to [0, \infty)$. The mapping G satisfies
$G(x, y, z) = 0$ if and only if $x = y = z$ for all $x, y, z \in X$;

(i) $0 < G(x, x, y)$ for all $x, y \in X$;
(ii) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;
(iii) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables);
(iv) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then G is called a G-metric on X and the pair (X, G) is called G-metric space. We combined the above two generalized metric spaces and defined the b_G-metric space as follows:

Definition 1.3 Let X be a non empty set and the mapping $b_G: X \times X \times X \to [0, \infty)$. The mapping b_G satisfies

(i) $b_G(x, y, z) = 0$ if and only if $x = y = z$ for all $x, y, z \in X$;
(ii) $b_G(x, x, y) \leq b_G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;
(iii) $b_G(x, y, z) = b_G(x, z, y) = b_G(y, z, x) = \cdots$ (symmetry in all three variables);
(iv) there exists a real numbers $s \geq 1$ such that
$$b_G(x, y, z) \leq s[b_G(x, a, a) + b_G(a, y, z)],$$
for all $x, y, z, a \in X$.

Then b_G is called a b_G metric on X. The ordered pair (X, b_G) is called b_G-metric space.

Example 1.4 Let $X = R$ and let
$$b_G(x, y, z) = \max\{|x - y|^2, |y - z|^2, |x - z|^2\}.$$
Then (X, b_G) is a b_G-metric space with the coefficient $s = 2$.

Example 1.5 Let $X = R^+$, $p > 1$ a constant and $b_G: X \times X \times X \to [0, \infty)$ be defined by
$$b_G(x, y, z) = \max\{(4x, y, z)^p, |4x - y - z|^p\},$$
for all $x, y, z \in X$. Then (X, b_G) is a b_G-metric space with$s > 1$.

Definition 1.6 Let (X, b_G) be a b_G-metric space, and let (x_n) be sequence of points in X, a point $x \in X$ is said to be the limit of the sequence (x_n), if $lim_{n,m\to\infty} b_G(x, x_n, x_m) = 0$.
In other words, if $x_n \to x$ in a b_G-metric space(X, b_G), it mean that for any given$\varepsilon > 0$, there exists $N \in N$ such that $b_G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$. In this case we say that the sequence (x_n) is b_G-converges to x.

Definition 1.7 Let (X, b_G) be a b_G- metric space, and a sequence (x_n) is called b_G-Cauchy, if, $\lim_{m,n,l\to\infty} b_G(x_n, y_m, z_l) = 0$ for all $x, y \in X$. Thus, that if $x_n \to x$ in a b_G-metric space (X, b_G), if for any $\varepsilon > 0$, there exists $N \in N$ such that $G(x_n, x_m, x_l) < \varepsilon$, for $n, m, l \geq N$.

Definition 1.8 Let (X, b_G) be a b_G- metric space. It is said to be complete if every b_G-Cauchy sequence converges in X.

Proposition 1.9 Let (X, b_G) be a b_G-metric space, then the following are equivalents:
(x_n) is b_G-convergent to x.

(i) $G(x_n, x_m, x) \to 0$, as $n \to \infty$.
(ii) $G(x_m, x, x) \to 0$, as $n \to \infty$.
(iii) $G(x_m, x_n, x) \to 0$, as $n \to \infty$.

Proposition 1.10 *Let* (X, b_G) be a b_G-metric space, then the following are equivalents.

(i) The sequence (x_n) is b_G-Cauchy.

(ii) For every$\varepsilon > 0$, there exists $N \in N$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$.

II. MAIN RESULTS

The first result of this paper is

Theorem 2.1 Let (X, b_G) be a complete b_G-metric space with $s \geq 1$ and let $T: X \to X$ be a mapping satisfying

$$G(Tx, Ty, Tz) \leq kG(x, y, z);$$

(2.1)

for all $x, y, z \in X$ and $k \in [0,1)$. Then T has a unique fixed point (say u i.e. $Tu = u$) and T is b_G-continuous at u.

Proof: Let $x_0 \in X$. Define a sequence (x_n) in X by $x_n = Tx_{n-1} = T^n x_0$. Consider

$$G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n)$$
$$\leq kG(x_{n-1}, x_n, x_n)$$
$$\leq k^2 G(x_{n-2}, x_{n-1}, x_{n-1})$$
$$\leq k^n G(x_0, x_1, x_1).$$

Moreover, for all $n, m \in N, n < m$ and using property (iv) of definition 1.3, we have

$$G(x_n, x_m, x_m) \leq s[G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_m, x_m)]$$
$$\leq s[k^n G(x_0, x_1, x_1) + G(x_{n+1}, x_m, x_m)]$$
$$\leq sk^n G(x_0, x_1, x_1) + s^2[G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_m, x_m)]$$
$$\leq sk^n G(x_0, x_1, x_1) + s^2 k^{n+1} G(x_0, x_1, x_1) + s^3[G(x_{n+2}, x_{n+3}, x_{n+3}) + G(x_{n+3}, x_m, x_m)]$$
$$\leq sk^n G(x_0, x_1, x_1) + s^2 k^{n+1} G(x_0, x_1, x_1) + s^3 k^{n+2} G(x_0, x_1, x_1) + s^3 G(x_{n+3}, x_m, x_m)]$$
$$\leq sk^n G(x_0, x_1, x_1) + s^2 k^{n+1} G(x_0, x_1, x_1) + s^3 k^{n+2} G(x_0, x_1, x_1) + \dots. + s^{m-1} k^{n+m-2} G(x_0, x_1, x_1) + s^{m-1} k^{n+m-1} G(x_0, x_1, x_1)$$
$$\leq sk^n[(1 + sk + (sk)^2 + (sk)^3 + \cdots + (sk)^{m-2}) + (sk)^{m-2} k] G(x_0, x_1, x_1)$$
$$\leq sk^n\left[\frac{1-(sk)^{n-(m-2)}}{(1-sk)} + (sk)^{m-2} k\right] G(x_0, x_1, x_1).$$

Since $k \in (0,1)$. Letting $m, n \to \infty$, $\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0$. Hence (x_n) is a Cauchy sequence in X. Since X is b_G complete, there exists $u \in X$ such that the sequence (x_n) is b_G-converges to u. Now we claim that u is fixed point of T i.e. $u = Tu$. Suppose $T(u) \neq u$. Consider

$$G(x_n, Tu, Tu) \leq s[G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, Tu, Tu]$$
$$\leq s[k^n G(x_0, x_1, x_1) + G(x_n, u, u)].$$

As $n \to \infty, x_n \to u$, we have

$$G(u, Tu, Tu) \leq sG(u, u, u).$$

This shows that $G(u, Tu, Tu)$ and hence$u = Tu$. This is contradiction to $T(u) = u$. Therefore u is a fixed point of$T$. Suppose $v \neq u$ and v is another fixed point of T i.e. $Tv = v$.

$$G(x_n, Tv, Tv) \leq s[G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, Tv, Tv)]$$
$$G(x_n, v, v) \leq s[k^n G(x_0, x_1, x_1) + G(x_n, v, v)].$$

As $n \to \infty, x_n \to u$ and

$$G(u, v, v) \leq sG(u, v, v)].$$

Since $s \geq 1$, we must have $G(u, v, v) = 0$. Hence $u = v$.

Now we will show that T is b_G-continuous at u. Let (y_n) be a sequence in X such that $\lim_{n\to\infty} y_n = u$. Consider

$$G(u, Ty_n, Ty_n) \leq kG(u, y_n, y_n).$$

But

$$G(y_n, Ty_n, Ty_n) \leq s[G(y_n, u, u) + G(u, Ty_n, Ty_n)]$$
$$\leq s[G(y_n, u, u) + kG(u, y_n, y_n)].$$

As $n \to \infty$, we have

$$G(u, Ty_n, Ty_n) \leq s[G(u, u, u) + kG(u, u, u)]$$
$$G(u, Ty_n, Ty_n) = 0.$$

Hence $Ty_n = u = Tu$. This show that T is b_G-continuous at u.

Corollary 2.2 Let (X, b_G) be a complete b_G-metric space with $s \geq 1$ and let $T: X \to X$ be a mapping satisfying for some $m \in N$

$$G(T^m(x), T^m(y), T^m(z)) \leq kG(x, y, z);$$

(2.2)

for all $x, y, z \in X$ and $k \in [0,1)$. Then T has a unique fixed point (say u, i.e. $Tu = u$) and T^m is b_G-continuous at u.

Theorem 2.3 Let (X, b_G) be a complete b_G-metric space with $s \geq 1$ and let $T: X \to$ X be a mapping satisfying

$$G(Tx, Ty, Tz) \leq k[G(x, Tx, Tx) + G(y, Ty, Ty)];$$

(2.3)

for all $x, y, z \in X$ and $k \in [0,1/2)$. Then T has a unique fixed point (say u i.e. $Tu = u$) and T is b_G-continuous at u.

Proof: Let $x_0 \in X$. Define a sequence (x_n) in X such that $x_n = Tx_{n-1} = T^n x_0$. Consider

$$G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n)$$
$$\leq k[G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})]$$

$$\leq kG(x_{n-1},x_n,x_n)+kG(x_n,x_{n+1},x_{n+1})$$
$$\leq \frac{k}{1-k}G(x_{n-1},x_n,x_n)$$
$$\leq \lambda G(x_{n-1},x_n,x_n)$$
$$\leq \lambda^n G(x_0,x_1,x_1),$$

where$\lambda=\frac{k}{1-k}<1$, since $k\in[0,1/2)$. Moreover, for all $n,m\in N, n<m$ and by property (iv) in definition 1.3, we have

$$G(x_n,x_m,x_m)\leq$$
$$s[G(x_n,x_{n+1},x_{n+1})+G(x_{n+1},x_m,x_m)]$$
$$\leq s[\lambda^n G(x_0,x_1,x_1)+G(x_{n+1},x_m,x_m)]$$
$$\leq s\lambda^n G(x_0,x_1,x_1)+s^2[G(x_{n+1},x_{n+2},x_{n+2})+G(x_{n+2},x_m,x_m)]$$
$$\leq s\lambda^n G(x_0,x_1,x_1)+s^2\lambda^{n+1}G(x_0,x_1,x_1)+s^3[G(x_{n+2},x_{n+3},x_{n+3})+G(x_{n+3},x_m,x_m)]$$
$$\leq s\lambda^n G(x_0,x_1,x_1)+s^2\lambda^{n+1}G(x_0,x_1,x_1)+s^3\lambda^{n+2}G(x_0,x_1,x_1)+s^3G(x_{n+3},x_m,x_m)]$$
$$\leq s\lambda^n G(x_0,x_1,x_1)+s^2\lambda^{n+1}G(x_0,x_1,x_1)+s^3\lambda^{n+2}G(x_0,x_1,x_1)+\ldots.+s^{m-1}\lambda^{n+m-2}G(x_0,x_1,x_1)+s^{m-1}\lambda^{n+m-1}G(x_0,x_1,x_1)$$
$$\leq s\lambda^n[(1+s\lambda+(s\lambda)^2+(s\lambda)^3+\ldots.+(s\lambda)^{m-2})+(s\lambda)^{m-2}\lambda]G(x_0,x_1,x_1)$$
$$\leq s\lambda^n[\frac{1-(s\lambda)^{n-(m-2)}}{(1-s\lambda)}+(s\lambda)^{m-2}\lambda]G(x_0,x_1,x_1).$$

Letting $m,n\to\infty$, we have $\lim_{n,m\to\infty}G(x_n,x_m,x_m)=0$. Hence (x_n) is a b_G-Cauchy sequence in X. Since X is complete, there exists $u\in X$ such that (x_n) is b_G converges to u. We claim that u is fixed point of T. Consider

$$G(x_n,Tu,Tu)\leq s[G(x_n,x_{n+1},x_{n+1})+G(x_{n+1},Tu,Tu)]$$

$$\leq s[\lambda^n G(x_0,x_1,x_1)+k[G(x_n,x_{n+1},x_{n+1})+G(u,Tu,Tu)]$$
$$\leq s\lambda^n(1+k)G(x_0,x_1,x_1)+skG(u,T(u),T(u)).$$

As $n\to\infty, x_n\to u$,

$$G(u,Tu,Tu)\leq skG(u,Tu,Tu).$$

This implies that $G(u,Tu,Tu)=0$ and hence $Tu=u$ i.e. u is a fixed point of T.

Suppose $v\neq u$ is another fixed point of T i.e. $T(v)=v$. Now

$$G(x_n,Tv,Tv)\leq s[G(x_n,x_{n+1},x_{n+1})+G(x_{n+1},Tv,Tv)]$$
$$G(x_n,v,v)\leq s[\lambda^n G(x_0,x_1,x_1)+k[G(x_n,x_{n+1},x_{n+1})+G(v,v,v)]$$
$$\leq s\lambda^n(1+k)G(x_0,x_1,x_1)+skG(v,v,v).$$

As $n\to\infty, x_n\to u$

$$G(u,v,v)\leq skG(u,v,v)].$$

Since $s\geq 1$, hence $G(u,v,v)=0$ and $u=v$.

Now we show that T is b_G-continuous at u. Let (y_n) be a sequence in X such that $\lim_{n\to\infty}y_n=u$. Consider

$$G(u,Ty_n,Ty_n)\leq k[G(u,u,u)+G(y_n,Ty_n,Ty_n)]$$
$$\leq kG(y_n,Ty_n,Ty_n).$$

But

$$G(y_n,Ty_n,Ty_n)\leq s[G(y_n,u,u)+G(u,Ty_n,Ty_n)]$$
$$\leq s[G(y_n,u,u)+kG(y_n,Ty_n,Ty_n)].$$

As $n\to\infty$,

$$G(u,Ty_n,Ty_n)\leq s[G(u,u,u)+kG(u,Ty_n,Ty_n)]$$
$$\leq skG(u,Ty_n,Ty_n).$$

Since $s\geq 1$ and $k\in[0,1/2)$, we must have $G(u,Ty_n,Ty_n)=0$. Hence $T(y_n)=u=T(u)$. This show that T is b_G-continuous at u.

Corollary 2.4 Let (X,b_G) be a complete b_G-metric space with $s\geq 1$ *and let* $T:X\to X$ be a mapping satisfying for some $m\in N$

$$G(T^m(x),T^m(y),T^m(z))\leq k[G(x,T^m(x),T^m(x))+G(y,T^m(y),T^m(y))] \quad (2.4)$$

for all $x,y,z\in X$ and $k\in[0,1/2)$. Then T has a unique fixed point (say u i.e. $Tu=u$) and T^m is b_G-continuous at u.

Theorem 2.5 *Let* (X,b_G) be a complete b_G-metric space with $s\geq 1$ and let $T:X\to X$ be a mapping satisfying

$$G(Tx,Ty,Tz)\leq k\max[G(x,y,z),G(x,Tx,Tx)+G(y,Ty,Ty)] \quad (2.5)$$

for all $x,y,z\in X$ and $k\in[0,1)$. Then T has a unique fixed point (say u i.e. $Tu=u$) and T is b_G-continuous at u.

Proof: Let $x_0\in X$. Define a sequence x_n in X such that $x_n=Tx_{n-1}=T^n x_0$. Consider

$$G(x_n,x_{n+1},x_{n+1})=G(Tx_{n-1},Tx_n,Tx_n)$$
$$\leq k\max[G(x_{n-1},x_n,x_n),G(x_{n-1},x_n,x_n),G(x_n,x_{n+1},x_{n+1})]$$
$$\leq kG(x_{n-1},x_n,x_n)$$
$$\leq k^n G(x_0,x_1,x_1).$$

Moreover for all $n,m\in N, n<m$ and by property (iv) in definition 1.3, we have

$G(x_n, x_m, x_m) \leq$
$s[G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_m, x_m)]$
$\leq s[k^n G(x_0, x_1, x_1) + G(x_{n+1}, x_m, x_m)]$
$\leq sk^n G(x_0, x_1, x_1) + s^2[G(x_{n+1}, x_{n+2}, x_{n+2}) +$
$G(x_{n+2}, x_m, x_m)]$
$\leq sk^n G(x_0, x_1, x_1) + s^2 k^{n+1} G(x_0, x_1, x_1) +$
$s^3[G(x_{n+2}, x_{n+3}, x_{n+3}) + G(x_{n+3}, x_m, x_m)]$
$\leq sk^n G(x_0, x_1, x_1) + s^2 k^{n+1} G(x_0, x_1, x_1) +$
$s^3 k^{n+2} G(x_0, x_1, x_1) + s^3 G(x_{n+3}, x_m, x_m)]$
$\leq sk^n G(x_0, x_1, x_1) + s^2 k^{n+1} G(x_0, x_1, x_1) +$
$s^3 k^{n+2} G(x_0, x_1, x_1) + \ldots + s^{m-1} k^{n+m-2} G(x_0, x_1,$
$x_1) + s^{m-1} k^{n+m-1} G(x_0, x_1, x_1)$
$\leq sk^n[(1 + sk + (sk)^2 + (sk)^3 + \ldots + (sk)^{m-2}) +$
$(sk)^{m-2} k] G(x_0, x_1, x_1)$
$\leq sk^n[\frac{1-(sk)^{n-(m-2)}}{(1-sk)} + (sk)^{m-2} k] G(x_0, x_1, x_1).$

Letting $m, n \to \infty$, we have $\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0$. Hence (x_n) is a b_G-Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that the sequence (x_n) is b_G converges to u. We claim that u is fixed point of T. Consider

$G(x_n, Tu, Tu) \leq$
$s[G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, Tu, Tu)]$
$\leq$
$s[k^n G(x_0, x_1, x_1) +$
$k\max[G(x_n, u, u), G(x_n, x_{n+1}, x_{n+1}), G(u, Tu, Tu)]$
$\leq sk^n G(x_0, x_1, x_1) + skG(u, Tu, Tu).$

As $n \to \infty, x_n \to u$, we have

$$G(u, Tu, Tu) \leq skG(u, Tu, Tu).$$

Since $s \geq 1$ and $k \in [0,1)$, we must have $G(u, Tu, Tu) = 0$ and hence $Tu = u$.

Suppose $v \neq u$ is another fixed point of T i.e. $Tv = v$. Consider

$G(x_n, Tv, Tv) \leq$
$s[G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, Tv, Tv)]$

$G(x_n, v, v) \leq$
$s[k^n G(x_0, x_1, x_1) +$
$k\max[G(x_n, v, v), G(x_n, x_{n+1}, x_{n+1}), G(v, v, v)]$
$\leq sk^n G(x_0, x_1, x_1) + skG(v, v, v).$

As $n \to \infty, x_n \to u$. Hence $G(u, v, v) = 0$, since $s \geq 1$. Thus $u = v$.

Now we show that T is b_G-continuous at u. Let (y_n) be a sequence in X such that $\lim_{n\to\infty} y_n = u$. Consider

$G(u, Ty_n, Ty_n)$
$\leq kmax[G(u, y_n, y_n), G(y_n, Ty_n, Ty_n), G(y_n, Ty_n, Ty_n)]$
$\leq kG(y_n, Ty_n, Ty_n).$

But

$$G(y_n, Ty_n, Ty_n) \leq s[G(y_n, u, u) + G(u, Ty_n, Ty_n)]$$
$$\leq s[G(y_n, u, u) + kG(y_n, Ty_n, Ty_n)].$$

Since $s \geq 1$ and $k \in [0,1)$, applying $n \to \infty$, we have

$$G(u, Ty_n, Ty_n) \leq s[G(u, u, u) + kG(u, Ty_n, Ty_n)]$$
$$\leq skG(u, Ty_n, Ty_n)$$

This implies $G(u, T(y_n), T(y_n)) = 0$. Hence $Ty_n = u = T(u)$. It shows that T is b_G-continuous at u.

Corollary 2.6 Let (X, b_G) be a complete b_G-metric space with $s \geq 1$ and let $T: X \to X$ be a mapping satisfying for some $m \in N$

$G(T^m x, T^m y, T^m z) \leq$
$k\max[G(x, y, z), G(x, T^m x, T^m x), G(y, T^m y, T^m y)]$; (2.6)

for all $x, y, z \in X$ and $k \in [0,1)$. Then T has a unique fixed point (say u i.e. $Tu = u$) and T^m is b_G-continuous at u.

Theorem 2.7 Let (X, b_G) be a complete b_G-metric space with $s \geq 1$ and let $T: X \to X$ be a mapping satisfying

$G(Tx, Ty, Tz) \leq$
$k\max[G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)]$ (2.7)

for all $x, y, z \in X$ and $k \in [0,1)$. Then T has a unique fixed point (say u i.e. $Tu = u$) and T is b_G-continuous at u.

Proof: Let $x_0 \in X$. Define a sequence x_n in X such that $x_n = Tx_{n-1} = T^n x_0$. Consider

$G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n)$
$\leq$
$k\max[G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1})]$
$\leq kG(x_{n-1}, x_n, x_n)$
$\leq k^n G(x_0, x_1, x_1).$

Moreover for all $n, m \in N, n < m$ and by property (iv) in definition 1.3, we have

$G(x_n, x_m, x_m) \leq$
$s[G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_m, x_m)]$
$\leq s[k^n G(x_0, x_1, x_1) + G(x_{n+1}, x_m, x_m)]$

$$\leq sk^n G(x_0,x_1,x_1) + s^2[G(x_{n+1},x_{n+2},x_{n+2}) + G(x_{n+2},x_m,x_m)]$$

$$\leq sk^n G(x_0,x_1,x_1) + s^2k^{n+1}G(x_0,x_1,x_1) + s^3[G(x_{n+2},x_{n+3},x_{n+3}) + G(x_{n+3},x_m,x_m)]$$

$$\leq sk^n G(x_0,x_1,x_1) + s^2k^{n+1}G(x_0,x_1,x_1) + s^3k^{n+2}G(x_0,x_1,x_1) + s^3G(x_{n+3},x_m,x_m)]$$

$$\leq sk^n G(x_0,x_1,x_1) + s^2k^{n+1}G(x_0,x_1,x_1) + s^3k^{n+2}G(x_0,x_1,x_1) + \ldots + s^{m-1}k^{n+m-2}G(x_0,x_1,x_1) + s^{m-1}k^{n+m-1}G(x_0,x_1,x_1)$$

$$\leq sk^n[(1 + sk + (sk)^2 + (sk)^3 + \ldots + (sk)^{m-2}) + (sk)^{m-2}k]G(x_0,x_1,x_1)$$

$$\leq sk^n[\tfrac{1-(sk)^{n-(m-2)}}{(1-sk)} + (sk)^{m-2}k]G(x_0,x_1,x_1).$$

Letting $m,n \to \infty$, we have $\lim_{n,m\to\infty} G(x_n,x_m,x_m) = 0$. Hence (x_n) is a b_G-Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that (x_n) is b_G-converges to u. We claim that u is fixed point of T. Consider

$$G(x_n,Tu,Tu) \leq s[G(x_n,x_{n+1},x_{n+1}) + G(x_{n+1},Tu,Tu)]$$

$$\leq s[k^n G(x_0,x_1,x_1) + k\max[G(x_n,x_{n+1},x_{n+1}),G(u,Tu,Tu),G(u,Tu,Tu)]$$

$$\leq sk^n G(x_0,x_1,x_1) + skG(u,Tu,Tu).$$

As $n \to \infty, x_n \to u$

$$G(u,Tu,Tu) \leq skG(u,Tu,Tu).$$

Since $s \geq 1$, we have $G(u,Tu,Tu) = 0$ and hence $Tu = u$ i.e. u is a fixed point of T.

Suppose $v \neq u$ is another fixed point of T i.e. $T(v) = v$. Consider

$$G(x_n,Tv,Tv) \leq s[G(x_n,x_{n+1},x_{n+1}) + G(x_{n+1},Tv,Tv)]$$

$$G(x_n,v,v) \leq s[k^n G(x_0,x_1,x_1) + k\max[G(x_n,x_{n+1},x_{n+1}),G(v,v,v),G(v,v,v)]$$

$$\leq sk^n G(x_0,x_1,x_1) + skG(v,v,v).$$

As $n \to \infty, x_n \to u$,

$$G(u,v,v) = 0,$$

since $s \geq 1$. This shows that $u = v$.

Now we show that T is b_G-continuous at u. Let (y_n) be a sequence in X such that $\lim_{n\to\infty} y_n = u$. Consider

$$G(u,Ty_n,Ty_n) \leq k\max[G(u,u,u),G(y_n,Ty_n,Ty_n),G(y_n,Ty_n,Ty_n)]$$

$$\leq kG(y_n,Ty_n,Ty_n).$$

But

$$G(y_n,Ty_n,Ty_n) \leq s[G(y_n,u,u) + G(u,Ty_n,Ty_n)]$$

$$\leq s[G(y_n,u,u) + kG(y_n,Ty_n,Ty_n)].$$

As $n \to \infty$,

$$G(u,Ty_n,Ty_n) \leq s[G(u,u,u) + kG(u,Ty_n,Ty_n)]$$

$$\leq skG(u,Ty_n,Ty_n).$$

Since $s \geq 1$ and$k \in [0,1)$, we have $G(u,Ty_n,Ty_n) = 0$. Hence $Ty_n = u = T(u)$. This shows that T is b_G-continuous at u.

Corollary 2.8 Let (X,b_G) be a complete b_G-metric space with $s \geq 1$ and let $T: X \to X$ be a mapping satisfying for some $m \in N$

$$G(T^m x,T^m y,T^m z) \leq k\max[G(x,T^m x,T^m x),G(y,T^m y,T^m y),G(z,T^m z,T^m z)] \quad (2.8)$$

for all $x,y,z \in X$ and $k \in [0,1)$. Then T has a unique fixed point (say u i.e. $Tu = u$) and T^m is b_G-continuous at u.

Theorem 2.9 Let (X,b_G) be a complete b_G-metric space with $s \geq 1$ and let $T: X \to X$ be a mapping satisfying

$$G(Tx,Ty,Tz) \leq a[G(x,Ty,Ty) + G(y,Tx,Tx)] \quad (2.9)$$

for all $x,y,z \in X$ and $as \in [0,1/2)$. Then T has a unique fixed point (say u i.e. $Tu = u$)and T is b_G-continuous at u.

Proof: Let $x_0 \in X$. Define a sequence x_n in X such that $x_n = Tx_{n-1} = T^n x_0$. Consider

$$G(x_n,x_{n+1},x_{n+1}) = G(Tx_{n-1},Tx_n,Tx_n)$$

$$\leq a[G(x_{n-1},x_{n+1},x_{n+1}) + G(x_n,x_n,x_n)]$$

$$\leq aG(x_{n-1},x_{n+1},x_{n+1})$$

$$\leq as[G(x_{n-1},x_n,x_n) + G(x_n,x_{n+1},x_{n+1})]$$

$$\leq \frac{as}{1-as}G(x_{n-1},x_n,x_n)$$

$$\leq kG(x_{n-1},x_n,x_n);\ k \in [0,1)$$

$$\leq k^n G(x_0,x_1,x_1).$$

Moreover for all $n,m \in N, n < m$ and by property (iv) of definition 1.3, we have

$$G(x_n,x_m,x_m) \leq s[G(x_n,x_{n+1},x_{n+1}) + G(x_{n+1},x_m,x_m)]$$

$\leq s[k^n G(x_0,x_1,x_1) + G(x_{n+1},x_m,x_m)]$

$\leq sk^n G(x_0,x_1,x_1) + s^2[G(x_{n+1},x_{n+2},x_{n+2}) + G(x_{n+2},x_m,x_m)]$

$\leq sk^n G(x_0,x_1,x_1) + s^2 k^{n+1} G(x_0,x_1,x_1) + s^3[G(x_{n+2},x_{n+3},x_{n+3}) + G(x_{n+3},x_m,x_m)]$

$\leq sk^n G(x_0,x_1,x_1) + s^2 k^{n+1} G(x_0,x_1,x_1) + s^3 k^{n+2} G(x_0,x_1,x_1) + s^3 G(x_{n+3},x_m,x_m)]$

$\leq sk^n G(x_0,x_1,x_1) + s^2 k^{n+1} G(x_0,x_1,x_1) + s^3 k^{n+2} G(x_0,x_1,x_1) + \ldots + s^{m-1} k^{n+m-2} G(x_0,x_1, x_1) + s^{m-1} k^{n+m-1} G(x_0,x_1,x_1)$

$\leq sk^n[(1 + sk + (sk)^2 + (sk)^3 + \ldots + (sk)^{m-2}) + (sk)^{m-2}k]G(x_0,x_1,x_1)$

$\leq sk^n[\frac{1-(sk)^{n-(m-2)}}{(1-sk)} + (sk)^{m-2}k]G(x_0,x_1,x_1).$

Letting $m,n \to \infty$, we have $\lim_{n,m\to\infty} G(x_n,x_m,x_m) = 0$. Hence (x_n) is a b_G-Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that (x_n) is b_G converges to u. Now we claim that u is fixed point of T. Consider

$G(x_n,Tu,Tu) \leq s[G(x_n,x_{n+1},x_{n+1}) + G(x_{n+1},Tu,Tu)]$

$\leq s[k^n G(x_0,x_1,x_1) + a[G(x_n,Tu,Tu) + G(u,x_{n+1},x_{n+1})]]$

$\leq sk^n G(x_0,x_1,x_1) + asG(u,Tu,Tu) + asG(u,x_{n+1},x_{n+1}).$

As $n \to \infty$, we have $x_n \to u$ and

$$G(u,Tu,Tu) \leq skG(u,Tu,Tu).$$

Since $s \geq 1$ and $k \in [0,1)$, we have $G(u,Tu,Tu) = 0$. Hence $Tu = u$. Therefore u is a fixed point of T.

Suppose $v \neq u$ is another fixed point of T i.e. $Tv = v$. Now

$G(x_n,Tv,Tv) \leq s[G(x_n,x_{n+1},x_{n+1}) + G(x_{n+1},Tv,Tv)]$

$G(x_n,v,v) \leq s[k^n G(x_0,x_1,x_1) + a[G(x_n,Tv,Tv) + G(v,x_n,x_n)]]$

$\leq sk^n G(x_0,x_1,x_1) + asG(x_n,v,v) + asG(v,x_n,x_n).$

As $n \to \infty$, we have $x_n \to u$ and

$$G(u,v,v) \leq asG(u,v,v).$$

Since $s \geq 1$ and $k \in [0,1)$, we must have $G(u,v,v) = 0$. Hence $u = v$.

Now we show that T is b_G-continuous at u. Let (y_n) be a sequence in X such that $\lim_{n\to\infty} y_n = u$. Consider

$$G(u,Ty_n,Ty_n) \leq a[G(u,Ty_n,Ty_n) + G(y_n,u,u)].$$

But

$G(y_n,Ty_n,Ty_n) \leq s[G(y_n,u,u) + a[G(u,Ty_n,Ty_n) + G(y_n,u,u)]]$

$\leq (1-a)sG(y_n,u,u) + asG(y_n,Ty_n,Ty_n).$

As $n \to \infty$, we have

$G(u,Ty_n,Ty_n \leq (1-a)sG(u,u,u) + asG(u,Ty_n,Ty_n)$

$\leq asG(u,Ty_n,Ty_n).$

Since $as < 1/2$, we must have $G(u,Ty_n,Ty_n) = 0$. Hence $Ty_n = Tu = u$. It shows that T is b_G-continuous at u.

Corollary 2.10 Let (X,b_G) be a complete b_G-metric space with $s \geq 1$ and let $T: X \to X$ be a mapping satisfying for some $m \in N$

$G(T^m x,T^m y,T^m z) \leq a[G(x,T^m y,T^m y) + G(y,T^m x,T^m x)]$ (2.10)

for all $x,y,z \in X$ and where $as \in [0,1/2)$. Then T has a unique fixed point (say u i.e. $Tu = u$) and T^m is b_G-continuous at u.

Example 2.11 Let $X = \mathrm{R}$ and define $G: R \times R \times R \to R$ by $G(x,y,z) = |x-y| + |y-z| + |x-z|$. Then (R,G) be a complete b_G-metric space. Let $T: R \to R$ defined by $T(x) = \frac{x}{3}$. Without loss of generality, we assume $x > y > z$. Then

(i)

$G(Tx,Ty,Tz) = |\frac{x}{3} - \frac{y}{3}| + |\frac{y}{3} - \frac{z}{3}| + |\frac{x}{3} - \frac{z}{3}|$

$= \frac{1}{3}[|x-y| + |y-z| + |x-z|]$

$\leq kG(x,y,z), k \in [0,1).$

(ii)

$G(Tx,Ty,Tz) = |\frac{x}{3} - \frac{y}{3}| + |\frac{y}{3} - \frac{z}{3}| + |\frac{x}{3} - \frac{z}{3}|$

$\leq \frac{x}{3} + \frac{y}{3} + \frac{x}{3}$

$\leq |x - \frac{x}{3}| + |y - \frac{y}{3}|$

$\leq \frac{1}{2}[2|x - \frac{x}{3}| + 2|y - \frac{y}{3}|]$

$\leq k[G(x,T(x),T(x)) + G(y,T(y),T(y))].$

(iii)

$G(Tx,Ty,Tz) = |\frac{x}{3} - \frac{y}{3}| + |\frac{y}{3} - \frac{z}{3}| + |\frac{x}{3} - \frac{z}{3}|$

$\leq \frac{x}{3} + \frac{x}{3}$

$\leq \frac{1}{2}[2|x - \frac{x}{3}|]$

$\leq k\max\{G(x,y,z), G(x,T(x),T(x)), G(y,T(y),T(y))\}.$

(iv)

$$G(Tx,Tx,Tx) = |\frac{x}{3} - \frac{y}{3}| + |\frac{y}{3} - \frac{z}{3}| + |\frac{x}{3} - \frac{z}{3}|$$

$$\leq \frac{x}{3} + \frac{x}{3}$$

$$\leq \frac{1}{2}[2|x - \frac{x}{3}|]$$

$$\leq$$

$k\max\{G(x,T(x),T(x)), G(y,T(y),T(y)), G(z,T(z),T(z))\}$.

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