Theorem B: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a polynomial of degree $n$, such that $a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0$, then all the zeros of $P(z)$ lie in the closed disk $|z| \leq 1$. In this paper we shall obtain some interesting extensions and generalizations of this result.

Theorem A: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a polynomial of degree $n$, such that $a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0$, then $P(z)$ has all its zeros in $|z| \leq 1$ in the literature [1-11], there exists various extensions and generalization of this theorem. Joya-etal [9] extended theorem A to polynomial whose coefficients are monotonic, but not necessarily non-negative by proving the following result.

Theorem B: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a polynomial of degree $n$, such that $a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0$. Then the zeros of $P(z)$ lie in $|z| \leq 1 + |a_n|/|a_0|$.

Recently Aziz and Zargar [2, 3] relaxed the hypothesis in various ways and in addition to other results they proved the following: Theorem C: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a polynomial of degree $n$, such that for some $k \geq 1, \ k a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0$, then the zeros of $P(z)$ lie in $|z| \leq 1 + |a_n|/|a_0|$.

Recently Aziz and Zargar generalized the above result by further relaxing the hypothesis. In fact they proved the following theorem.

Theorem D: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a polynomial of degree $n$, if for some positive numbers $k = n$ and $\rho = 1$, $k a_n \geq a_{n-1} \geq \ldots \geq \rho a_0 > 0$, then the zeros of $P(z)$ lie in the closed disk $|z + k - 1| \leq k + 2a_0 a_n / (1 - \rho)$.

Main results In this Paper we prove some generalizations of the Enestrom-Kakeya, theorem by relaxing the hypothesis in various ways and we shall present interesting generalizations of theorem A, B, C. In this direction we prove the following theorem.

**Theorem 1**: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_p z^p + a_{p-1} z^{p-1} + \ldots + a_1 z + a_0$ be a polynomial of degree $n$, if for some $k \geq 1, a_n \geq a_{n-1} \geq \ldots \geq a_p \geq a_{p-1} \geq \ldots \geq a_1 \geq a_0 > 0$, then all the zeros of $P(z)$ lie in the closed disk $|z + k - 1| \leq k + 2(\lambda - 1) a_p / a_n$.

**Remark 1**: For $k = 1$ this result reduces to theorem A, also applying the above result to the polynomial $(t z)$, we get the following **Corollary 1**: Let $P(t z) = a_n t^n z^n + a_{n-1} t^{n-1} z^{n-1} + \ldots + a_p t^p z^p + \ldots + a_1 z + a_0$ be a polynomial of degree $n$, if for some positive number $\lambda = \frac{k+1}{k}$, then all zeros of $P(t z)$ lie in $|t z| \leq 1 + \left(\frac{2(k-1) a_p}{a_n a_0}\right)^{\frac{1}{k-1}}$.

**Theorem 2**: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_p z^p + \ldots + a_1 z + a_0$ be a polynomial of degree $n$, for some $k \geq 1$, $a_n \geq a_{n-1} \geq \ldots \geq a_p \geq a_{p-1} \geq \ldots \geq a_1 \geq a_0$, then all the zeros of $P(z)$ lie in the closed disk $|z + k - 1| \leq k + 2(\lambda - 1) a_p / a_n$.

**Remark 2**: For $k = 1$, the above result reduces to theorem B. If the coefficients are non-negative, then this result reduces to theorem A above. Hence this result is more general **Theorem 3**: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_p z^p + \ldots + a_1 z + a_0$ be a polynomial of degree $n$, for some $k \geq 1$ and $\lambda \geq 1$, then the zeros of $P(z)$ lie in $|z + k - 1| \leq k + 2(\lambda - 1) a_p / a_n$.

**Remark 3**: For $\lambda = 1$ this result reduces theorem C, the corresponding result. If the non-negativity condition is relaxed, then the above result reduces to the following:
**Theorem 4:** Let \( P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_p z^p + \sum a_i z + a_0 \) be a polynomial of degree \( n \), for some \( k \geq 1 \) and \( \lambda \geq 1 \) \( k a_n \geq a_{n-1} \geq \cdots \geq a_p \geq \lambda a_p \geq a_{p-1} + \cdots + a_1 \geq a_0 \), then \( P(z) \) has all its zeros in the circle \(|z| \leq 1 + \frac{2(\lambda - 1)a_p - a_0 + |a_0|}{|a_n|}\).

**Proof of Theorem 1:** Consider the polynomial

\[
F(z) = (1 - z)P(z) = \sum a_i z^i = -a_n z^n + (a_n - a_{n-1}) z^{n-1} + \cdots + (a_{p+1} - a_p) z^{p+1} + \sum a_i z + a_0.
\]

Hence

\[
|F(z)| \geq \left| \frac{(a_n - a_{n-1}) z^{n-1} + \cdots + (a_{p+1} - a_p) z^{p+1}}{z^{n-1} + \cdots + z + 1} \right|.
\]

If \(|z| > 1 + \frac{2(\lambda - 1)a_p - a_0 + |a_0|}{|a_n|}\),

\[
F(z) = -a_n z^n + (a_n - a_{n-1}) z^{n-1} + \cdots + (a_{p+1} - a_p) z^{p+1} + \sum a_i z + a_0
\]

Therefore all the zeros of \( F(z) \), whose modulus is greater than 1 lie in the circle \(|z| \leq 1 + \frac{2(\lambda - 1)a_p - a_0 + |a_0|}{|a_n|}\). But those zeros of \( F(z) \), whose modulus is less than or equal to 1 already satisfy above inequality and all the zeros of \( P(z) \) are also the zeros of \( F(z) \), therefore all zeros of \( P(z) \) lie in the circle \(|z| \leq 1 + \frac{2(\lambda - 1)a_p - a_0 + |a_0|}{|a_n|}\).

**Proof of Theorem 2:** Consider the polynomial

\[
F(z) = (1 - z)P(z) = \sum a_i z^i = -a_n z^n + (a_n - a_{n-1}) z^{n-1} + \cdots + (a_{p+1} - a_p) z^{p+1} + \sum a_i z + a_0.
\]

This gives

\[
|F(z)| \geq \left| \frac{(a_n - a_{n-1}) z^{n-1} + \cdots + (a_{p+1} - a_p) z^{p+1}}{z^{n-1} + \cdots + z + 1} \right|.
\]

If \(|z| > 1 + \frac{2(\lambda - 1)a_p - a_0 + |a_0|}{|a_n|}\),

\[
F(z) = -a_n z^n + (a_n - a_{n-1}) z^{n-1} + \cdots + (a_{p+1} - a_p) z^{p+1} + \sum a_i z + a_0
\]

Therefore all the zeros of \( F(z) \), whose modulus is greater than 1 lie in the circle \(|z| \leq 1 + \frac{2(\lambda - 1)a_p - a_0 + |a_0|}{|a_n|}\). But those zeros of \( F(z) \), whose modulus is less than or equal to 1 already satisfy above inequality and all the zeros of \( P(z) \) are also the zeros of \( F(z) \), therefore all zeros of \( P(z) \) lie in the circle \(|z| \leq 1 + \frac{2(\lambda - 1)a_p - a_0 + |a_0|}{|a_n|}\).
Proof of theorem 3:

Consider the polynomial

\[ F(z) = (1 - z)P(z) \]

\[ = -a_nz^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \ldots + (a_{p+1} - a_p)z^{p+1} + (a_p - a_{p-1})z^p + \ldots + (a_1 - a_0)z + a_0 \]

Therefore all the zeros of \( F(z) \), whose modulus is greater than 1 lie in the circle

\[ |z| < 1 + \frac{2(k-1)a_p-a_{k-1}a_0}{|a_n|}. \]

But those zeros of \( F(z) \), whose modulus is less than or equal to 1, already satisfy the above inequality. Since all the zeros of \( P(z) \) are also the zeros of \( F(z) \), therefore all zeros of \( P(z) \) lie in the circle

\[ |z| \leq 1 + \frac{2(k-1)a_p-a_{k-1}a_0}{|a_n|}. \]

Proof of theorem 3: Consider the polynomial

\[ F(z) = (1 - z)P(z) \]

\[ = -a_nz^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \ldots + (a_{p+1} - a_p)z^{p+1} + (a_p - a_{p-1})z^p + \ldots + (a_1 - a_0)z + a_0 \]

Therefore all the zeros of \( F(z) \), whose modulus is greater than 1 lie in the circle

\[ |z| < 1 + \frac{2(k-1)a_p-a_{k-1}a_0}{|a_n|}. \]

But those zeros of \( F(z) \), whose modulus is less than or equal to 1, already satisfy the above inequality and all the zeros of \( P(z) \) are also zeros of \( F(z) \), therefore all zeros of \( P(z) \) lie in the circle

\[ |z| \leq 1 + \frac{2(k-1)a_p-a_{k-1}a_0}{|a_n|}. \]
Proof of theorem 4: Consider the polynomial

\[ F(z) = (1-z)P(z) = (1-z)(a_n z^n + a_{n-1} z^{n-1} + \ldots + a_p z^p + \ldots + a_1 z + a_0) \]

\[ = -a_n z^{n+1} + (a_n - a_{n-1}) z^n + (a_{n-1} - a_{n-2}) z^{n-1} + \ldots + (a_{p+1} - a_p) z^{p+1} + (a_p - a_{p-1}) z^p + \ldots + (a_1 - a_0) z + a_0 \]

Hence,

\[ |F(z)| = \left| -a_n z^{n+1} + (a_n - a_{n-1}) z^n + (a_{n-1} - a_{n-2}) z^{n-1} + \ldots + (a_{p+1} - a_p) z^{p+1} + (a_p - a_{p-1}) z^p + \ldots + (a_1 - a_0) z + a_0 \right| \]

\[ \geq |a_n||z^n||z + k - 1| - \left\{ \left( \frac{|ka_n|}{|z|} - 1 \right) |a_n - a_{n-1}| |z|^n + \ldots + |a_{p+1} - a_p| |z|^{p+1} \right\} \]

\[ \geq |a_n||z^n||z + k - 1| - \left\{ \left( \frac{|ka_n|}{|z|} - 1 \right) |a_n - a_{n-1}| |z|^n + \ldots + |a_{p+1} - a_p| |z|^{p+1} \right\} \]

This gives

\[ |F(z)| \geq |a_n||z^n||z + k - 1| - \left\{ \left( \frac{|ka_n|}{|z|} - 1 \right) |a_n - a_{n-1}| |z|^n + \ldots + |a_{p+1} - a_p| |z|^{p+1} \right\} \]

\[ \geq |a_n||z^n||z + k - 1| - \left( \left( \frac{|ka_n|}{|z|} - 1 \right) |a_n - a_{n-1}| |z|^n + \ldots + |a_{p+1} - a_p| |z|^{p+1} \right) \]

For \( |z| > 1 \), we have

\[ |F(z)| \geq |a_n||z^n||z + k - 1| - \left( \left( \frac{|ka_n|}{|z|} - 1 \right) |a_n - a_{n-1}| |z|^n + \ldots + |a_{p+1} - a_p| |z|^{p+1} \right) \]

\[ |F(z)| \geq |a_n||z^n||z + k - 1| - \left( \left( \frac{|ka_n|}{|z|} - 1 \right) |a_n - a_{n-1}| |z|^n + \ldots + |a_{p+1} - a_p| |z|^{p+1} \right) \]

\[ |F(z)| \geq |a_n||z^n||z + k - 1| - \left( \left( \frac{|ka_n|}{|z|} - 1 \right) |a_n - a_{n-1}| |z|^n + \ldots + |a_{p+1} - a_p| |z|^{p+1} \right) \]

\[ |F(z)| \geq |a_n||z^n||z + k - 1| - \left( \left( \frac{|ka_n|}{|z|} - 1 \right) |a_n - a_{n-1}| |z|^n + \ldots + |a_{p+1} - a_p| |z|^{p+1} \right) \]

But those zeros of \( F(z) \), whose modulus is less than or equal1, already satisfy the above inequality and all the zeros of \( P(z) \) are also zeros of \( F(z) \), therefore all zeros of \( P(z) \) lie in the circle

\[ |z + k - 1| \leq k + \frac{2|a_n| + 2|a_{p+1} - a_p|}{|a_n|} \]

II. REFERENCES


