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Construction and Enumeration of STS19, 21, 23

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Abstract:
The higher order Steiner system is very complex in nature. Various methodologies can be used to construct and enumerate the STS for order of 19, 21, 23. The various combinatory theory and design theory can be used for the enumeration of the desired STS. Here, in this paper, we have presented a detailed method of the enumeration, methodology and construction of the STS systems. Detailed numerical analysis has been done for the enumeration of the STS of order 23. The algorithm discussed will here generate the total possible STS combinations. Finally, construction methodology has also been presented. Graphical construction has been discussed with the reference to the kirkman systems. The properties of the STS system equations have been discussed. The various theorems have been represented using the order 9 subsystems. Results have shown the steps of the enumeration and combinations of various pairs showing the properties of STS of the order 23.

Keywords: Steiner triple systems, Auto-morphism, non-isomorphism

I. INTRODUCTION:

A Steiner triple system [3] of order v is a set of 3-element subsets, called blocks, of a v -set of points, such that every pair of points occurs in exactly one block. For an STS(v), standard counting arguments show that each point must occur in exactly $r=(v-1)/2$ blocks and that the triple system consists of exactly $b=v\,(v-1)/6$ blocks where r and b are integers we get necessary conditions for the existence of an STS(v).

For $v>3$ an STS(v) exists if and only if either $v=1(\mathrm{mod}\,6)$ or $v=3(\mathrm{mod}\,6)$. Two STS are isomorphic if thereexists a bijection between the point sets that maps blocks onto blocks; such a bijection is an isomorphism. An auto-morphism[5][17] of a triple system is an isomorphism of the triple system onto itself. The auto-morphism group of the triple system consists of all of its automorphisms. The number of pairwise non-isomorphic STS (v) is denoted by N(v). N(v)grows exponentially as proved .The first few nonzero values of the function N(v)are $N(3)=1,\ N(7)=1,\ N(9)=1,\ N(13)=2, and\ N(15)=80$

Theoretically, assuming a collection of Q objects, on average at least log2Q bits are required to uniquely represent (identify) an object in the collection. Concatenating the compressed representations of all the objects, at least Qlog2Q bits, or approximately 46 gigabytes for Q = N(19), are required

STS of order 19 [19]:

Let (X,B) be an STS(v). A subsystem of order w (denoted sub-STS(w)) is a pair (Y,V) which is an STS(w) in its own right. An STS (19) can contain subsystems of orders 1 or 3 (trivially), 7, or 9. There are at least 2450 non-isomorphic STS (19) which contain 3 sub-STS (7) intersecting in a point.It is shown that there are exactly 284457 non-isomorphic STS (19) containing sub-STS (9). As each STS (19) was generated, we stored the relevant information on tape: the random seed used to generate the system, the two invariants, and two Boolean flags indicating the presence of subsystems.

Enumeration of STS (19) with the sub-order of system 9

There are exactly 284457 non-isomorphic STS (19) which contain a sub-STS (9)[19], and at least 2111230 which do not have isomorphs. The estimated population is the solution N to the equation $(1-R/N)=(1-1/N)/S$. This equation can easily be solvedby Newton's method, giving N = 3.54 X 108. Although N is huge, it is probably too low, for two reasons. First, the populationwe are estimating is that of invariants, and not non-isomorphic STS (19). It canhappen, and does, that non-isomorphic STS (19) have the same invariants. We know of no way to estimate the probability of this occurring. Second, we construct STS (19) by means of a pseudo-random number generator with a period of 230. Soour population is restricted to those 230(= 109) STS(19) which can be constructedfrom the particular pseudo-random number generator we used. We can obtain an estimate in another way, by considering the STS(19) with a sub-STS(9).We have noted that there are exactly 284457 such STS seeds. Of the 2117600seeds of STS we constructed, 46 out of those STS seeds, contained a sub-STS(9). Taking ratios, we get an estimate ofN = 284457/46 X 2117600 = 1.309 X 109. It may be that our hill-climbing algorithmis less likely to produce STS (19) containing sub-STS (9) than a truly randomalgorithm; so this estimate may be too high.

CALCULATION PARAMETERS

- The point set $\{1,2,3,........,v\}$ is used for all STS(v), and the vertex set $\{1,2,.........,b\}$ is used for all graphs of order b
- The symmetric group on $\{1,2,.........,b\}$ is denoted by S_b. The auto-morphism group of a graph G is denoted by Aut (G)

- Similarly, the auto-morphism group of a triple system B is denoted by Aut(B)
- Block Graphs (Linear Graphs) : A graph whose vertices are in one-to-one correspondence with the blocks of triple system, with two vertices joined by an edge if and only if the corresponding blocks have nonempty intersections

PASCH CONFIGURATION:

- A Pasch configuration (or fragment or quadrilateral) in an STS is a set of four blocks and six points of the form $\{u,v,w\},\{w,x,y\},\{u,x,z\},\{v,y,z\}$
- An STS is said to be anti-Pasch if it does not contain a Pasch configuration
- We define collections of the blocks where a pair of distinct collections of blocks $(T1,T2)$ is said to be mutually $t-$ balanced if each $t-$ element subset of the base set V contained in precisely the same number of blocks of $T1$as of $T2$.Each collection $(T1,T2)$ is then referred to as a trade.
- The Pasch configuration is the smallest trade that can occur in a Steiner triple system. If $T1$ is the collection $\{a,b,c\},\{a,y,z\},\{x,b,z\},\{x,y,c\}$, then, by replacing each triple with its complement, a collection $T1$ $\{x,y,z\},\{x,b,c\},\{a,y,c\},\{a,b,z\}$ is obtained which contains precisely the same pairs as $T1$. This transformation is known as a Pasch switch, and when applied to a Steiner triple system yields another, usually non-isomorphic, Steiner triple system

THEORMS OF GRAPHS: STS

- **Theorem -1:** A Regular Graph with v vertices and degree k is said to be strongly regular graph[17][20] if it has two integers $\lambda\, and\, \mu$, having two adjacent neighbors λ ,and two non-adjacent vertices having μ common neighbors represented as srg(v,k,λ,μ) .
- Block graph of an STS(v) is a strong regular graph:

The converse of this theorem holds on the condition that v is large enough. This result was obtained by Bose, for $v > 67$ .The following theorem is one of the central building blocks of our classification approach

- **Theorem – 2:**For every $v >= 19$,every STS(v) can be reconstructed up to isomorphism from its block graph

The proof of Theorem contains an explicit algorithm [19] for constructing anSTS from a strongly regular graph. We do not, however, need such an algorithmhere, since every such graph will be explicitly constructed from an STS, and thetransformation between these objects is therefore known

- **Theorem – 3:**For every $v >= 19$. Two STS(v) are isomorphic if and only if their block graphs are isomorphic
- **Theorem-4:** For $v >= 19$, a block graph of a $STS(v)$ contains exactly v r -cliques. The vertices of an r -clique in the block graph clearly correspond to a set of r blocks with pairwise nonempty intersection[17]

A short case-by-case analysis showsthat a set of blocks with pairwise nonempty intersection in an STS(v) has size at most 7 unless the blocks share a common point. Since each point occurs in r blocks, and no two blocks contain the same point pair, there are exactly v sets of r blocks with pairwise nonempty intersection when $r > 7$, that is, $v >= 19$

- **Theorem-5:** For $v >= 19$, the auto-morphism group of an STS(v)and the auto-morphism group of a corresponding block graph are isomorphic. Label the blocks of an STS(v) and consider the associatedblock graph G. Clearly, the block automorphism group of the STS is asubgroup of Aut(G)
- **Theorem-6:** Each block of an STS(v)occurs in exactly $(v-3)(v-3)(v\text{^}2 -12v+99)/16$ sets of four blocks that have pairwise non-empty intersection but do not form a pasch configuration
- For $v >= 19$, the function ∇_p is well defined and constitutes a vertex invariant for the set of all block graphs derived from STS(v)

CONSTRUCTION OF STS

- **STAGE-I :**Preprocessing stage in which the seeds for the main search are determined
- **STAGE-II:** The second stage consists of determining all extensions of each seed to an STS(19), STS(21), STS(23) and rejecting isomorphs
- The correspondence between Steiner triple systems and strongly regular graphs given by Theorem is very useful from an algorithmic point of view since we can alternate between representations and use the best representation for the task at hand. The core of our algorithm is an efficient exact cover algorithm for constructing STS(19).
- Isomorph-free generation is achieved using the block graph representation and **nauty** supplemented with the Pasch configuration vertex invariant.
- The details of our approach are as follows. The construction process has two stages. The first stage is a preprocessing stage in which the seeds for the main search are determined. The second stage consists of determining all extensions of each seed to an STS(19) and rejecting isomorphs

The problem of finding all extensions of a seed sub-design to an STS(19) is that of finding all solutions to an instance of exact cover. In the exact cover problem, we are given a set and a collection of its subsets; the task is to cover the set with given subsets so that each element of the set is covered exactly once

- In the preprocessing stage, we fix the first block, $\{1,2,3\}$ .Construct all pairwise non-isomorphic designs consisting of 3-element blocks that intersect the first block so that the total number of blocks is 25 (r = 9 for an STS(19)) and no pair in $\{1,2,\ldots 19\}$ occurs in more than one block.
- We generate a table for all possible incident matrix up to isomorphism
- The structure of seeds needs to determine two parts of the sub-seed design: A and B
- Part A can be completed with combinatorial arguments
- Part-B can be calculated by carrying out isomorphic rejection in each matix for Part-A
- For each completion we perform isomorph rejection against a stored collection of orbit representatives
- Each design is encoded as a vertex-colored bipartite graph in which vertices of one color correspond to the points, vertices of another color correspond to the blocks, and edges encode the incidence relation between points and blocks

- Pair-wise non-isomorphic m-block seed sub-designs are obtained in this way
- The problem of finding all extensions of a seed sub-design to an STS(19) is that of finding all solutions to an instance of exact cover
- In the exact cover problem, we are given a set and a collection of its subsets; the task is to cover the set with given subsets so that each element of the set is covered exactly once
- We combine the data of complete search with previous results to calculate number of pair-wise non-isomorphic STS(19)
- First, we take the representatives from the isomorphism classes of STS(19)
- Orbit-stabilizer theorom gives total number of STS(19)

Constructing and extending seeds.

In the preprocessing stage, we construct all pairwisenon-isomorphic designs consisting of 3-element blocks that intersect block so that the total number of blocksis 25.

A backtrack search with isomorphrejection is carried out. Actually, the A part can be completed up to isomorphismusing combinatorial arguments; there are only seven such completions. From left to right these correspond to the seven partitions,4 + 4 + 4 + 4; 4 + 4 + 8; 4 + 6 + 6; 4 + 12; 6 + 10; 8 + 8; 16;of 16 into even integers greater than or equal to 4. Each completion corresponds to a 1-factor of the complete graph K16 that is disjoint from the 1-factor in columns. The union of two such 1-factors is a 2-regular graph consistingof even-length cycles only; up to isomorphism these correspond to the partitions above For each completion we perform isomorph rejection withnauty against a stored collection of orbit representatives. Each design is encodedas a vertex-colored bipartite graph in which vertices of one color correspond to thepoints, vertices of another color correspond to the blocks, and edges encode theincidence relation between points and blocks .In total, 14;648 pairwisenon-isomorphic 25-block seed sub-designs are obtained in this way.

Then the orbit-stabilizer theorem gives as the total number of STS(19), STS(21), STS(23).

Again, the orbit-stabilizer theorem can be usedto get the total number of STS(19), STS(21),STS(23) . Since we know the number of STS(19) for *which* $j\,\mathrm{Aut}(Bi)j>1$ and all values are known after the search, it is straightforward to determine the number of STS(19) STS(21),STS(23) with trivial auto-morphism group.

ISOMPORPH REJECTION

The most involved part of the algorithm is the eliminationof isomorphic STS (19), STS (21), and STS (23) from consideration. There are three issues that need to be addressed. First, the main search must be conducted in parallel because of the considerable resource requirements. This presents a difficulty since the parallel runs shouldpreferably be independent of each other, whereby no comparisons between isomorphism class representatives encountered in distinct runs are allowed. Second, thesearch is to be conducted in part on computers that do not have enough mainmemory to store the millions of isomorphism class representatives potentially encounteredas extensions of a single seed sub-design. Third, isomorphism testing must be fast since there are in the order of, as we now know, tested for isomorphism. All of the above difficulties are essentially solved by a recent algorithm framework for iso-morph-freeexhaustive generation. Our basic isomorph rejection strategy is to use nauty to compute the canonicallabeling and automorphism orbits of the block graph of a generated STS(19),STS(21),STS23). Toenable parallelization we must guarantee that algorithm runs performed on differentseeds sub-designs do not output isomorphic STS(19). This can be accomplished byusing the output of nauty to test that a generated STS(19) originates from thecorrect parent seed subdesign.

Theorem: Let sub-seeds B and B0 be two generated STS(19), STS(21),STS(23) that pass the test. . If Band B0 are isomorphic, then they have been constructed by extending the same seedsubdesign S --- (7)

Moreover, there exists an automorphism of S that is an isomorphismof B onto B0

Theorem: For every isomorphism class of STS (19), there exists an STS(19)that is an extension of a seed sub design and that passes the test -- (8)

This can be extended to STS(21) and STS(23).

We must still perform isomorph rejection on those STS(19)that have been generated from the same seed sub-design.If the automorphism group of the seed sub-design is large, then the further testwe employ is simply a hash table query to see whether the canonically labeled block graph

II. BLOCK CONSTRUCTION OF STS:

A Steiner Triple System, denoted by STS(v), is a pair (S,T) consisting of a set S with v elements, and a set Tconsisting of triples of S (called blocks) such thatevery pair of elements of S appear together in aunique triple of T.

Quasi-groups:

A **Quasigroup** $(S,\otimes)$ is a set S together with a binary operation ($\otimes$) such that:

1. The operation is closed (i.e., a $\otimes$ b)
2. Given a,b the equations

i) a $\otimes$ x = b and

ii) y $\otimes$ a = b

have unique solutions for x and y.

A simple example of a finite quasigroup is given by the set $\{0,1,2\}$ with the operation define by a $\otimes$ b = 2a+b+1 where the operations on the right are the usual multiplication and addition modulo 3.

Latin Squares and Quasi-groups

Theorem 1.1: The multiplication table of a quasigroup is a Latin square--------------------- (9)

Where the entry which occurs in the r-th rowand s-th column is the product $a_r \otimes a_s$ of the elements a_r and a_s . Ifthe same entry occured twice in the r -th row, say in the s -thand r -th columns so that a_{rs} = a_{rt} = b, have two solutions to the equation $a_r \otimes x = b$, in contradiction to thequasigroup axiom. Similarly, if the same entry occurred twice inthe s-th column, we would have two solutions to the equation $y \otimes a_s = c$ for some c . We conclude that each element of thequasigroup occurs exactly once in each row and column, and sothe unbordered multiplication table (which is an $n \times n$ array) is alatin square

Theorem 1.2: There is no commutative idempotent latin square of even Order -------------- (10)

For any $n = 2k + 1$, there exists a commutative idempotent latin square of order n.

III. RESULTS & DISCUSSIONS

The Bose construction of an STS $(6n + 3)$ for any natural number $(Q, \otimes)$, utilizes a commutative idempotent quasi-group $(Q, \otimes)$ of order $2n + 1$ .The set S consists of the $6n + 3$ ordered pairs of $Q \times \{0, 1, 2\}$ and the triples T are of two types:

Type 1: $\{(i,0), (i,1), (i,2)\}$

Type 2: $\{(i, k), (j, k), (i \quad j, k + 1 (\mathrm{mod}\ 3))\}$

There are $2n + 1$ triples of type 1 and $3(2n + 1)(2n)/2 = 6n^2 + 3n$ triples of type 2.

Thus,

$| T | = 6n^2 + 5n + 1 = (6n + 3)(6n + 2)/6 = v(v - 1)/6$

Let (a, b) and (c, d) be distinct elements of S . If $a = c$ then this pair is in a triple of type 1.

If $b = d$, the pair is in a triple of type 2. Now, either $d = b + 1 (\mathrm{mod}\ 3)$ or. In the first case, let x be the unique solution of $a \otimes x = c$ in Q. The triple containing the pair is thus $d = b - 1 (\mathrm{mod}\ 3)$ $\{(a, b), (x, b), (c, d)\}$. In the second case, let y be the unique solution of $y \otimes c = a$ in Q. The triple is then $\{(y,d), (c, d), (a, b)\}$

IV. FINAL CONSTRUCTION:

This construction of an STS($6n + 1$) starts with a set Sconsisting of the 6 n ordered pairs of Q $\times\{0,1,2\}$, where (Q,⊗)is a commutative half-idempotent quasigroup of order 2 n . To describe the triples we assume that the quasigroup Qhas symbols $\{1,2,...,2n\}$. The triples are then:

Type 1: $\{(i,0), (i,1), (i,2)\}$

Type 2: $\{ \Theta, (i, k), (n + i, k - 1\ \mathrm{mod}\ 3)\}$

Type 3: $\{(i, k), (j, k), (i \otimes j, k + 1 (\mathrm{mod}\ 3))\}$

Suppose (a, b) and (c, d) are a pair of elements of S.

The triplecontaining the pair is thus $\{(a, b), (x, b), (a, d)\}$ In the second case, let y be the unique solution of y ⊗a = a in Q. Again, the triple is then $\{(y,d), (a, d), (a, b)\}$

RESULTS: STS (19)

10 12 16
10 13 18
1 11 12
11 13 19
12 14 17
1 13 17
14 15 19
1 15 18
17 18 19
2 10 11
2 12 13
2 14 18
2 16 19
2 3 5
1 2 4
2 6 8
2 7 15
2 9 17
3 13 15
3 16 17
1 3 19
3 4 18
3 6 10
3 7 12
3 8 14
3 9 11
4 10 14
4 11 15
4 13 16
4 5 17
4 6 7
4 8 19
4 9 12
5 10 19
5 11 14
5 12 15
5 16 18
5 6 13
1 5 7
5 8 9
6 11 16
6 12 19
1 6 14
6 15 17
6 9 18
7 10 17
7 11 18
7 14 16
7 8 13
7 9 19
8 10 15
8 11 17
8 12 18
1 8 16
1 9 10
9 13 14
9 15 16

RESULTS: (STS 21)

10 12 21
10 13 16
10 14 17
1 10 20
1 11 12
11 13 14
12 15 20

1 13 19
1 14 15
14 18 19
15 17 21
1 16 17
16 18 21
19 20 21
2 11 21
2 12 13
2 14 20
2 15 16
2 17 18
2 3 5
1 2 4
2 6 19
2 7 8
2 9 10
3 10 15
3 11 16
3 12 14
3 17 19
3 18 20
1 3 21
3 4 13
3 6 8
3 7 9
4 10 19
4 11 20
4 15 18
4 5 17
4 6 21
4 7 14
4 8 16
4 9 12
5 11 19
5 12 16
5 13 20
1 5 18
5 6 10
5 7 21
5 8 14
5 9 15
6 11 17
6 12 18
6 13 15
6 14 16
1 6 7
6 9 20
7 10 11
7 12 17
7 13 18
7 15 19
7 16 20
8 10 18
8 11 15
8 12 19
8 13 21
8 17 20
1 8 9
9 11 18
9 13 17
9 14 21
9 16 19

RESULTS: (STS 23)
21 10 2
23 10 5
4 9 2
10 13 16
10 14 17
1 10 20
1 11 12
11 13 14
12 15 20
1 13 19
1 14 15
14 18 19
15 17 21
1 16 17
16 18 21
19 20 21
2 11 21
2 12 13
2 14 20
2 15 16
2 17 18
2 3 5
1 2 4
2 6 19
2 7 8
2 9 10
3 10 15
3 11 16
3 12 14
3 17 19
3 18 20
1 3 21
3 4 13
3 6 8
3 7 9
4 10 19
4 11 20
4 15 18
4 5 17
4 6 21
4 7 14
4 8 16
4 9 12
5 11 19
5 12 16
5 13 20
1 5 18
5 6 10
5 7 21
5 8 14
5 9 15
6 11 17
6 12 18
6 13 15
6 14 16
1 6 7
6 9 20

7 10 11
7 12 17
7 13 18
7 15 19
7 16 20
21 19 11
23 9 2

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