ISSN XXXX XXXX © 2016 IJESC

Research Article Volume 6 Issue No. 12

Geodetic Dominating Sets and Geodetic Dominating Polynomials of Cycles

A.Vijayan[1], N. Jaspin Beaula[2]
Associate Professor[1], Assistant Professor[2]
Department of Mathematics
Nesamony Memorial Christian College, Marthandam, Tamil Nadu, India[1]
BWDA Polytechnic College, vendalicode, Tamil Nadu, India[2]

Abstract:
Let G = (V, E) be a simple graph. A set $S \subseteq V$ is a dominating set of G, if every vertex in V – S is adjacent to atleast one vertex S. Let $D_g(C_n,i)$be the family of geodetic dominating sets of the graph C_n with cardinality 'i'. Let $d_g(C_n, i) = |D_g(C_n, i)|$. In this paper, we obtain a recursive formula for $d_g(C_n, i)$.Using the recursive formula, we construct the polynomial, $D_g(C_n, x) = \sum_{i=\left\lceil \frac{n+2}{3} \right\rceil}^{n} d_g(C_n, i)\, x^i$ which we call geodetic dominating polynomial of C_n and obtain some properties of this polynomial.

Keywords: geodetic domination set, geodetic domination number, geodetic domination polynomial.

I. INTRODUCTION

Let G = (V,E) be a simple graph of order n. For any vertex $v \in V$, the open neighborhood of v is the set, $N(v) = \{u \in V/ uv \in E\}$ and the closed neighborhood of v is $N[V] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a dominating set of G,if $N[S] = V$ or equivalently in every vertex in V – S is adjacent to atleast one vertex in S. The domination number of a graph G is defined as the minimum size of a dominating set of vertices in G and is denoted by $\gamma(G)$. A cycle can be defined as a closed path and is denoted by C_n. A subset S of vertices in a graph G is called a geodetic set if every vertex not in S lies on a shortest path between two vertices from S. A subset D of the set of vertices is called geodetic dominating set if D is both a geodetic and dominating set.

1.1. Definition

Let $D_g(C_n, i)$ be the family of geodetic dominating sets of the graph C_n with cardinality 'i' and let $d_g(C_n, i) = |D_g(C_n, i)|$. We call the polynomial $D_g(C_n,x) = \sum_{i=\left\lceil \frac{n+2}{3} \right\rceil}^{n} d_g(C_n, i)\, x^i$, the geodetic domination polynomial of the graph C_n.

In the next section, we construct the families of the geodetic dominating sets of the cycles by recursive method.

As usual we use $\lfloor x \rfloor$ for the largest integer less than or equal to x and $\lceil x \rceil$ for the smallest integer greater than equal to x. Also we denote the set $\{1,2,\ldots,n\}$ by $[n]$ throught out this paper.

II. GEODETIC DOMINATING SETS OF CYCLES

Let $Dg(C_n, i)$ be the family of geodetic dominating sets of C_n with cardinality i ,we investigate the geodetic dominating sets of C_n , we need the following lemma to prove our main results in this section.

2.1. Lemma

$$\gamma_g(C_n) = \left\lceil \frac{n+2}{3} \right\rceil$$

By Lemma 2.1 and the definition of geodetic domination number, one has the following lemma:

2.2. Lemma

$D_g(C_n, x) = \Phi$ if and only if $i > n$ or $i < \left\lceil \frac{n+2}{3} \right\rceil$.

A simple path is a path in which all internal vertices have degree two.

2.3. Lemma

Let C_n, $n \geq 3$ be the cycle with $|V(C_n)| = n$.

(i) If $D_g(C_{n-1}, i-1) = D_g(C_{n-3}, i-1) = \Phi$, then $D_g(C_{n-2}, i-1) = \Phi$.

(ii) If $D_g(C_{n-1}, i-1) \neq \Phi$ and $D_g(C_{n-3}, i-1) \neq \Phi$, then $D_g(C_{n-2}, i-1) \neq \Phi$.

(iii) If $D_g(C_{n-1}, i-1) = \Phi$, $D_g(C_{n-2}, i-1) = \Phi$;and $D_g(C_{n-3}, i-1) = \Phi$ then $D_g(C_n, i) = \Phi$

Proof

(i) If $D_g(C_{n-1}, i-1) = \Phi$ and $D_g(C_{n-3}, i-1) = \Phi$, then

$$i-1 < \left\lceil \frac{n+1}{3} \right\rceil \text{ or } i-1 > n-1 \text{ and}$$

$$i-1 < \left\lceil \frac{n-1}{3} \right\rceil \text{ or } i-1 > n-3$$

$$\text{Therefore, } i-1 < \left\lceil \frac{n-1}{3} \right\rceil \text{ or } i-1 > n-1$$

$$\text{Therefore, } i-1 < \left\lceil \frac{n}{3} \right\rceil \text{ or } i-1 > n-2 \text{ holds.}$$

$D_g(C_{n-1}, i-1) = \Phi$.

(ii) If $D_g(C_{n-1}, i-1) \neq \Phi$ and $D_g(C_{n-3}, i-1) \neq \Phi$, then

$$\left\lceil \frac{n+1}{3} \right\rceil \leq i-1 \leq n-1 \text{ and}$$

$$\left\lceil \frac{n-1}{3} \right\rceil \leq i-1 \leq n-3$$

$$\text{Therefore, } \left\lceil \frac{n+1}{3} \right\rceil \leq i-1 \leq n-3$$

$$\text{Therefore, } \left\lceil \frac{n}{3} \right\rceil \leq \left\lceil \frac{n+1}{3} \right\rceil \leq i-1 \leq n-3 < n-2$$

$$\text{Therefore, } \left\lceil \frac{n}{3} \right\rceil \leq i-1 < n-2$$

Hence, $D_g(C_{n-2}, i-1) \neq \Phi$.

(ii) Since, $D_g(C_{n-1}, i-1) = \Phi$, $D_g(C_{n-2}, i-1) = \Phi$ and $D_g(C_{n-2}, i-1) = \Phi$,

$$\text{We have, } i-1 < \left\lceil \frac{n+1}{3} \right\rceil \text{ or } i-1 > n-1,$$

$$i-1 < \left\lceil \frac{n}{3} \right\rceil \text{ or } i-1 > n-2 \text{ and}$$

$$i-1 < \left\lceil \frac{n-1}{3} \right\rceil \text{ or } i-1 > n-3$$

$$\text{Hence, } i-1 < \left\lceil \frac{n-1}{3} \right\rceil \text{ or } i-1 > n-1$$

$$\text{Therefore, } i < \left\lceil \frac{n-1}{3} \right\rceil + 1 \text{ or } i > n$$

$$\text{Therefore, } i < \left\lceil \frac{n+2}{3} \right\rceil \text{ or } i > n$$

Therefore, $D_g(C_n, i) = \Phi$.

2.4. Theorem

Let C_n, $n \geq 3$ be the cycle with $|v(C_n)| = n$.

Suppose that $D_g(C_n, i) \neq \Phi$ then

(i) $D_g(C_{n-1}, i-1) = D_g(C_{n-2}, i-1) = \Phi$ and $D_g(C_{n-3}, i-1) \neq \Phi$ if and only if $n = 3k-2$ and $i = k$ for some $k \in N$.

(ii) $D_g(C_{n-2}, i-1) = \Phi$; $D_g(C_{n-3}, i-1) = \Phi$ and $D_g(C_{n-1}, i-1) \neq \Phi$ if and only if $i = n$.

(iii) $D_g(C_{n-1}, i-1) = \Phi$; $D_g(C_{n-2}, i-1) \neq \Phi$; $D_g(C_{n-3}, i-1) \neq \Phi$ if and only if $n = 3k$ and $i = \left\lceil \frac{3k+3}{3} \right\rceil$ for some $k \in N$.

(iv) $D_g(C_{n-1}, i-1) \neq \Phi$; $D_g(C_{n-2}, i-1) \neq \Phi$ and $D_g(C_{n-3}, i-1) = \Phi$ if and only if $i = n-1$

(v) $D_g(C_{n-1}, i-1) \neq \Phi$; $D_g(C_{n-2}, i-1) \neq \Phi$; $D_g(C_{n-3}, i-1) \neq \Phi$ if and only if $\left\lceil \frac{n+1}{3} \right\rceil + 1 \leq i \leq n$.

Proof

(i) Since, $D_g(C_{n-1}, i-1) = D_g(C_{n-2}, i-1) = \Phi$,

By Lemma 2.2, $i-1 > n-1$ or $i-1 < \left\lceil \frac{n+1}{3} \right\rceil$ and

$$i-1 > n-2 \text{ or } i-1 < \left\lceil \frac{n}{3} \right\rceil.$$

$$\text{Therefore, } i-1 < \left\lceil \frac{n}{3} \right\rceil \text{ or } i-1 > n-1$$

If $i-1 > n-1$, then $i > n$

Therefore, $D_g(C_n, i) = \Phi$ which is a contradiction.

$$\text{Therefore, } i-1 < \left\lceil \frac{n}{3} \right\rceil.$$

$$\text{Therefore, } i < \left\lceil \frac{n}{3} \right\rceil + 1.$$

Also, since, $D_g(C_{n-3}, i-1) \neq \Phi$,

$$\text{We have, } \left\lceil \frac{n-1}{3} \right\rceil \leq i-1 \leq n-3.$$

$$\text{Hence, } \left\lceil \frac{n-1}{3} \right\rceil + 1 \leq i < \left\lceil \frac{n}{3} \right\rceil + 1.$$

This is true only when $n = 3k-2$ and $i = k$ for some $k \in N$.

Conversely, assume $n = 3k-2$ and $i = k$, for some $k \in N$.

$$\text{By Lemma 2.2, } \gamma_g(C_n) = \left\lceil \frac{n+2}{3} \right\rceil.$$

$D_g(C_{n-1}, i-1) = D_g(C_{3k-2-1}, K-1) = \Phi$,

$$\text{Since, } k-1 < \left\lceil \frac{3k-3+2}{3} \right\rceil = \left\lceil \frac{3k-1}{3} \right\rceil$$

Similarly, $D_g(C_{n-2}, i-1) = \Phi$.

$D_g(C_{n-3}, i-1) = D_g(C_{3k-2-3}, k-1)$

$= D_g(C_{3k-5}, k-1)$.

$$\text{Since, } k-1 \geq \left\lceil \frac{3k-5+2}{3} \right\rceil = \left\lceil \frac{3k-3}{3} \right\rceil, D_g(C_{n-3}, i-1) \neq \Phi.$$

Hence, $D_g(C_{n-1}, i-1) = \Phi$; $D_g(C_{n-2}, i-1) = \Phi$ and $D_g(C_{n-3}, i-1) \neq \Phi$.

(ii) Since, $D_g(C_{n-2}, i-1) = \Phi$ and $D_g(C_{n-3}, i-1) = \Phi$,

by Lemma 2.2, $i-1 > n-2$ or $i-1 < \left\lceil \frac{n-1}{3} \right\rceil$.

If $i-1 < \left\lceil \frac{n-1}{3} \right\rceil$ then $i-1 < \left\lceil \frac{n+1}{3} \right\rceil$.

Therefore, by Lemma 3.3,

$D_g(C_{n-1}, i-1) = \Phi$ which is a contradiction.

So we have $i-1 > n-2$.

That is $i > n - 1$.

Therefore, $i \geq n$.

Since, $D_g(C_{n-1}, i-1) \neq \Phi$, $\left\lceil \frac{n+1}{3} \right\rceil < i - 1 \leq n - 1$

Therefore, $i \leq n$.

Hence, $i = n$.

Conversely, if $i = n$, then $D_g(C_{n-2}, i-1) = D_g(C_{n-2}, n-1) = \Phi$

$D_g(C_{n-3}, i-1) = D_g(C_{n-3}, n-1) = \Phi$

And $D_g(C_{n-1}, i-1) = D_g(C_{n-1}, n-1) = [n] \neq \Phi$,

since, $|D_g(C_{n-1}, n-1)| = 1$.

(iii) Since, $D_g(C_{n-1}, i-1) = \Phi$, by Lemma 2.2, $i - 1 > n - 1$.

Therefore, $i - 1 > n - 2$.

Therefore, $D_g(C_{n-2}, n-1) = \Phi$ and $D_g(C_{n-3}, n-1) = \Phi$ which is a contradiction.

Therefore, $i - 1 < \left\lceil \frac{n+1}{3} \right\rceil$

Therefore, $i < \left\lceil \frac{n+1}{3} \right\rceil + 1$

Since, $D_g(C_{n-2}, i-1) \neq \Phi$ and $D_g(C_{n-3}, i-1) \neq \Phi$,

We have, $\left\lceil \frac{n}{3} \right\rceil \leq i - 1 \leq n - 2$ and $\left\lceil \frac{n-1}{3} \right\rceil \leq i - 1 \leq n - 3$

Therefore, $\left\lceil \frac{n}{3} \right\rceil \leq i - 1 \leq n - 3$.

Hence, $\left\lceil \frac{n}{3} \right\rceil + 1 \leq i \leq \left\lceil \frac{n+1}{3} \right\rceil + 1$.

This holds only if $n = 3k$ and $i = k + 1 = \left\lceil \frac{3k+3}{3} \right\rceil$ for some $k \in N$

Conversely, assume $n = 3k$ and $i = \left\lceil \frac{3k+3}{3} \right\rceil$, then by Lemma 2.2,

$D_g(C_{n-1}, n-1) = \Phi$; $D_g(C_{n-2}, n-1) = \Phi$ and

$D_g(C_{n-3}, n-1) \neq \Phi$

(iv) Assume $D_g(C_{n-1}, n-1) \neq \Phi$; $D_g(C_{n-2}, n-1) \neq \Phi$ and

$D_g(C_{n-3}, n-1) \neq \Phi$.

Since, $D_g(C_{n-3}, n-1) = \Phi$, by Lemma 2.2,

$i - 1 > n - 3$ or $i - 1 < \left\lceil \frac{n-1}{3} \right\rceil$.

Since, $D_g(C_{n-2}, n-1) \neq \Phi$, $\left\lceil \frac{n}{3} \right\rceil \leq i - 1 \leq n - 2$.

Therefore, $i - 1 < \left\lceil \frac{n-1}{3} \right\rceil$ is not possible.

$i - 1 > n - 3$

Therefore, $i - 1 \geq n - 2$

But $i - 1 \leq n - 2$

$i - 1 = n - 2$

$i = n - 1$

Conversely, suppose $i = n - 1$, then

$D_g(C_{n-1}, i-1) = D_g(C_{n-1}, n-2) \neq \Phi$,

$D_g(C_{n-2}, i-1) = D_g(C_{n-2}, n-2) \neq \Phi$.

But, $D_g(C_{n-3}, i-1) = D_g(C_{n-3}, n-2) = \Phi$.

By Lemma 2.2, $D_g(C_{n-1}, i-1) \neq \Phi$; $D_g(C_{n-2}, i-1) \neq \Phi$ and

$D_g(C_{n-3}, i-1) = \Phi$.

(v) Assume $D_g(C_{n-1}, i-1) \neq \Phi$; $D_g(C_{n-2}, i-1) \neq \Phi$ and

$D_g(C_{n-3}, i-1) \neq \Phi$.

Then, by Lemma 2.2, we have $\left\lceil \frac{n+1}{3} \right\rceil \leq i - 1 \leq n - 1$;

$\left\lceil \frac{n}{3} \right\rceil \leq i - 1 \leq n - 2$ and $\left\lceil \frac{n-1}{3} \right\rceil \leq i - 1 \leq n - 3$

Therefore, $\left\lceil \frac{n+1}{3} \right\rceil \leq i - 1 \leq n - 3$

Also, since, $D_g(C_n, i) \neq \Phi$, we have $\left\lceil \frac{n+2}{3} \right\rceil \leq i \leq n$

But $\left\lceil \frac{n+1}{3} \right\rceil + 1 \leq i \leq n - 2$

Therefore, $\left\lceil \frac{n+1}{3} \right\rceil + 1 \leq i \leq n - 2$

Conversely, suppose that $\left\lceil \frac{n+1}{3} \right\rceil + 1 \leq i \leq n - 2$

Therefore, $\left\lceil \frac{n+1}{3} \right\rceil \leq i - 1 \leq n - 1$ and

$\left\lceil \frac{n}{3} \right\rceil \leq i - 1 \leq n - 2$ and $\left\lceil \frac{n-1}{3} \right\rceil \leq i - 1 \leq n - 3$

From these, we obtain that $D_g(C_{n-1}, i-1) \neq \Phi$,

$D_g(C_{n-2}, i-1) \neq \Phi$ and $D_g(C_{n-3}, i-1) \neq \Phi$.

2.5. Theorem

For every $n \geq 6$ and $i \geq \left\lceil \frac{n+2}{3} \right\rceil$,

(i) If $D_g(C_{n-1}, i-1) = D_g(C_{n-2}, i-1) = \Phi$ and $D_g(C_{n-3}, i-1) \neq \Phi$, then $D_g(C_n, i) = \{\{1,2,\ldots n-5,n\}, \{1,3,\ldots,n-4,n\}, \{1,4,\ldots,n-3,n\}, \{1,5,\ldots,n-2,n\}, \{1,6,\ldots,n-1,n\}\}$.

(ii) If $D_g(C_{n-2}, i-1) = D_g(C_{n-3}, i-1) = \Phi$ and $D_g(C_{n-1}, i-1) \neq \Phi$, then $D_g(C_n, n) = \{1,2,3,\ldots,n\}$.

(iii) If $D_g(C_{n-1}, i-1) \neq \Phi$; $D_g(C_{n-2}, i-1) \neq \Phi$ and $D_g(C_{n-3}, i-1) = \Phi$, then $D_g(C_n, n-1)$

$= \{[n] - \{x\} / x \in [n]\}$

(iv) If $D_g(C_{n-1}, i-1) \neq \Phi$; $D_g(C_{n-2}, i-1) \neq \Phi$ and $D_g(C_{n-3}, i-1) = \Phi$, then $D_g(C_n, i)$

$= \{ x_1 \cup \{n\} / x_1 \in D_g(C_{n-1}, i-1)\} \cup$

$\{ x_2 \cup \{n-1\} / x_2 \in D_g(C_{n-2}, i-1)\} \cup$

$\{ x_3 \cup \{n-2\} / \in D_g(C_{n-3}, i-1)\}$

Proof

(i) Since, $D_g(C_{n-1}, i-1) = D_g(C_{n-3}, i-1) = \Phi$ and $D_g(C_{n-3}, i-1) \neq \Phi$, by Lemma(2.4) (i), $n = 3k - 2$, $i = k$ for some $k \in N$

The sets {1,2,…n – 5,n},{1,3,…,n – 4 ,n}, (1,4,…,n – 3 ,n},{1,5,…,n – 2 ,n},{1 ,6,…,n – 1 ,n}
Are the only geodetic dominating sets of minimum cardinality $\left\lceil \frac{n+2}{3} \right\rceil$.

(ii) Since $D_g(C_{n-2}, i-1) = \Phi$; $D_g(C_{n-3}, i-1) = \Phi$ and $D_g(C_{n-1}, i-1) \neq \Phi$; by Lemma 2.4(2), i = n
Therefore, $D_g(C_n, n) = \{\{1,2,3,\ldots,n\} = \{[n]\}$

(iii) If $D_g(C_{n-1}, i-1) \neq \Phi$; $D_g(C_{n-2}, i-1) \neq \Phi$ and $D_g(C_{n-3}, i-1) = \Phi$, then, by Lemma 2.4,
i = n –1
Therefore, $D_g(C_n, i) = D_g(C_n, n-1)$
$= \{[n] - \{x\} / x \in [n]\}$

(iv) Let $x_1 \in D_g(C_{n-1}, i-1)$, so atleast one vertex labeled n – 1 , n – 2 or n – 3 is in x_1.
If n – 1 , n– 2 or n – 3 $\in x_1$,then $x_1 \cup \{n\} \in D_g(C_n, i)$.
Let $x_2 \in D_g(C_{n-2}, i-1)$, then n – 2 , n – 3 or n – 4 in x_2 .
If n – 2 , n– 3, n – 4 $\in x_2$ then $x_2 \cup \{n-1\} \in D_g(C_n, i)$.
Now Let $x_3 \in D_g(C_{n-3}, i-1)$, then n – 3 , n – 4 or n – 5 is in x_3 .
If n – 3 or n– 4or n – 5 $\in x_3$ then $x_3 \cup \{n-2\} \in D_g(C_n, i)$
Thus, we have $\{x_1 \cup \{n\} / x_1 \in D_g(C_{n-1}, i-1)\} \cup$
$\{x_2 \cup \{n-1\} / x2 \in D_g(C_{n-2}, i-1)\} \cup$
$\{x_3 \cup \{n-2\} / x_3 \in D_g(C_{n-3}, i-1) \subseteq D_g(C_n, i)$ ….. (2.5)

If n $\in$Y,then Y = $x_1 \cup \{n\}$ for some $x_1 \in D_g(C_{n-1}, i-1)$
If n $\notin$Y and n – 1 $\in$Y, then Y = x2 $\cup$ {n–1} for some $x_2 \in D_g(C_{n-2}, i-1\}$
If n $\notin$Y , n – 1 $\notin$Y and n – 2 $\in$Y,then Y = $x_3 \cup$ {n– 2} for some $x_3 \in D_g(C_{n-3}, i-1\}$
So $D_g(C_n, i) \subseteq \{x_1 \cup \{n\} / x_1 \in D_g(C_{n-1}, i-1)\} \cup$
$\{x_2 \cup \{n-1\} / x_2 \in D_g(C_{n-2}, i-1)\} \cup$
$\{ x_3 \cup \{n-2\} / x_3 \in D_g(C_{n-3}, i-1)$ ……(2.6)

From (2.5) and (2.6) we have
$D_g(C_n, i) = \{x_1 \cup \{n\} / x_1 \in D_g(C_{n-1}, i-1)\} \cup$
$\{x_2 \cup \{n-1\} / x_2 \in D_g(C_{n-2}, i-1)\} \cup$
$\{ x_3 \cup \{n-2\} / x_3 \in D_g(C_{n-3}, i-1)$

III. GEODETIC DOMINATION POLYNOMIAL OF A CYCLE

Let $D_g(C_n, x) = \sum_{i=\left\lceil \frac{n+2}{3} \right\rceil}^{n} D_g(C_n, i)\, x^i$ be the geodetic domination polynomial of a cycle C_n. In this section, we derive an expression for $D_g(C_n, x)$.

3.1. Theorem

a) If $D_g(C_n, x)$ is the family of geodetic dominating sets with cardinality i of C_n , then
$d_g(C_n, i) = d_g(C_{n-1}, i-1) + d_g(C_{n-2}, i-1) + d_g(C_{n-3}, i-1)$ where $d_g(C_n, i) = |D_g(C_n, i)|$.

b) For every n ≥ 8,
$D_g(C_n, x) = x\,[D_g(C_{n-1}, x) + D_g(C_{n-2}, x) + D_g(C_{n-3}, x)$ with initial values
$D_g(C_3, x) = x^3$,
$D_g(C_4, x) = x^4 + 2x^3$,
$D_g(C_5, x) = x^5 + 3x^4 + 3x^3$,
$D_g(C_6, x) = x^6 + 4x^5 + 6x^4 + 4x^3$,
$D_g(C_7, x) = x^7 + 5x^6 + 10x^5 + 9x^4 + 5x^3$.

Proof

a) Suppose (iv) of theorem 2.5 holds.
From (iv), we have
$D_g(C_n, i) = \{x_1 \cup \{n\} / x_1 \in D_g(C_{n-1}, i-1)\} \cup$
$\{x_2 \cup \{n-1\} / x_2 \in D_g(C_{n-2}, i-1)\} \cup$
$\{x_3 \cup \{n-2\} / x_3 \in D_g(C_{n-3}, i-1)$
Therefore, $|D_g(C_n, i)| = |D_g(C_{n-1}, i-1)| \cup |D_g(C_{n-2}, i-1)| \cup |D_g(C_{n-3}, i-1)|$
Therefore, $d_g(C_n, i) = d_g(C_{n-1}, i-1) + d_g(C_{n-2}, i-1) + d_g(C_{n-3}, i-1)$
Hence, we have the theorem.

b) $\Sigma\, d_g(C_n, i)\, x^i = \Sigma\, d_g(C_{n-1}, i-1)\, x^i + \Sigma\, d_g(C_{n-2}, i-1)\, x^i + \Sigma\, d_g(C_{n-3}, i-1)\, x^i$
$\Sigma\, d_g(C_n, i)\, x^i = x\,\Sigma\, d_g(C_{n-1}, i-1)\, x^{i-1} + x\Sigma\, d_g(C_{n-2}, i-1)\, x^{i-1} + x\Sigma\, d_g(C_{n-3}, i-1)\, x^{i-1}$
$\Sigma\, d_g(C_n, i)\, x^i = x[\Sigma\, d_g(C_{n-1}, i-1)\, x^{i-1} + \Sigma\, d_g(C_{n-2}, i-1)\, x^{i-1} + \Sigma\, d_g(C_{n-3}, i-1)\, x^{i-1}]$
$D_g(C_n, x) = x[d_g(C_{n-1}, x) + d_g(C_{n-2}, x) + d_g(C_{n-3}, x)]$
with initial values
$D_g(C_3, x) = x^3$,
$D_g(C_4, x) = x^4 + 2x^3$,
$D_g(C_5, x) = x^5 + 3x^4 + 3x^3$,
$D_g(C_6, x) = x^6 + 4x^5 + 6x^4 + 4x^3$,
$D_g(C_7, x) = x^7 + 5x^6 + 10x^5 + 9x^4 + 5x^3$.
We obtain $d_g(C_n, i)$ for 3 ≤ n ≤14 as shown in table 1

Table 1

i	3	4	5	6	7	8	9	10	11	12	13	14
n												
3	1											
4	2	1										
5	3	3	1									
6	4	6	4	1								
7	5	9	10	5	1							
8	0	12	18	15	6	1						
9	0	9	27	32	21	7	1					
10	0	5	30	55	52	28	8	1				
11	0	0	26	75	102	79	36	9	1			
12	0	0	14	83	162	185	114	45	10	1		
13	0	0	5	70	213	326	292	158	55	11	1	
14	0	0	0	45	228	477	590	442	212	66	12	1

In the following theorem, we obtain some properties of $d_g(C_n, i)$

3.2. Theorem

The following properties hold for the coefficients of $D_g(C_n, x)$;
(i) $d_g(C_n, n) = 1$ for every n ≥ 3.
(ii) $d_g(C_n, n-1) = n - 2$ for every n ≥ 4.

(iii) $d_g(C_n, n-2) = \frac{(n-2)(n-3)}{2}$ for every $n \geq 5$.

(iv) $d_g(C_n, n-3) = \frac{n^3 - 9n^2 + 20n + 12}{6}$ for every $n \geq 6$.

(v) $d_g(C_n, n-4) = \frac{1}{24}$ $[n^4 - 14n^3 + 47n^2 + 134n - 720]$ for every $n \geq 6$.

Proof

(i) Since $d_g(C_n, n = \{[n]\}$,we have the result.

(ii) Since $d_g(C_n, n-1) = \{[n] - \{x\}/x \in [x]\}$,

We have $d_g(C_n, n-1) = n - 2$.

(iii)To prove $d_g(C_n, n-2) = \frac{(n-2)(n-3)}{2}$ for every $n \geq 5$.

We apply induction on n. When n =5,

L.H.S = $d_g(C_5, 5-2) = d_g(5,3) = 3$(from the table) and

R.H.S = $\frac{(n-2)(n-3)}{2} = \frac{(5-2)(5-3)}{2} = 3$

Therefore, the result is true for n =5.

Now suppose that the result is true for all natural numbers less than n, and we prove it for 'n'

By Theorem 3.1, we have,

$$d_g(C_n, n-2) = d_g(C_{n-1}, n-3) + d_g(C_{n-2}, n-3) + d_g(C_{n-3}, n-3)$$

$$= \frac{1}{2}[(n\text{-}1)^2 - 5(n\text{-}1) + 2] + n\text{–}2 + 1$$

$$= \frac{1}{2}[n^2 - 2n + 1\text{–}5n\text{+}1\text{+} 2 + 2n - 4\text{+} 25(n\text{–}1) = 2]$$

$$= \frac{1}{2}[n^2 - 5n + 6]$$

Hence the result is true for all n.

(iv) To prove $d_g(C_{n\text{-}3}, n-3) = \frac{n^3 - 9n^2 + 20n + 12}{2}$

For every $n \geq 6$, we apply induction on n.

When n = 6, L.H.S = $d_g(6, 3)$=5 (from the table)

and R.H.S = $\frac{1}{6}(6^3 - 9 \times 6^2 + 20 \times 6 + 12]$

= 4.

Therefore, the result is true for n =4.

Now suppose that the result is true for all numbers less than n and we prove it for 'n'.

By theorem 3.1,we have

$$d_g(C_n, n-3) = d_g(C_{n-1}, n-4) + d_g(C_{n-2}, n-4) + d_g(C_{n-3}, n-4)$$

$$= \frac{1}{6}[(n\text{–}1)^3 - 9(n\text{–}1)^2 + 20(n\text{–}1) + 12]$$

$$+ \frac{1}{2}[(n\text{–}1)^2 - 5(n\text{–}1) + 2] + n\text{–}3$$

$$= \frac{1}{6}[n^2 - 3n^2 + 3n\text{–}1\text{–} 9n^2 + 18n - 9\text{+} 20n\text{–} 20 + 12]$$

$$+ \frac{1}{2}[n^2 - 9n + 6] + n - 3$$

$$= \frac{1}{6}[n^3 - 12n^2 + 41n\text{–}18] + \frac{1}{2}[n^2 - 9n + 6] + n - 3$$

$$= \frac{n^3 - 12n^2 + 41n + 18 + 3n^2 - 27n + 48 + 6n - 18 + 12}{6}$$

$$= \frac{n^3 - 9n^2 + 20n + 12}{6}$$

Hence, the result is true for all n.

v) To prove $d_g(C_n, n-4) = \frac{1}{24}[n^4 - 14n^3 + 47n^2 + 134n - 720]$ for every $n \geq 7$.we apply by induction on n.

When n = 7,L.H.S = $D_g(7, 3)$ = 5(from the table).

And R.H.S = $\frac{1}{24}$ $[7^4 - 14(7)^3 + 47(7)^2 + 134(7) - 720] = 5$.

Therefore the result is true for n = 7.

Now, suppose that the result is true for all natural numbers less than n ,and we prove it for n.By theorem 3.1,we have

$$d_g(C_n, n-4) = d_g(C_{n-1}, n-4) + d_g(C_{n-2}, n-4) + d_g(C_{n-3}, n-4)$$

$$= \frac{1}{24}[(n\text{–}1)^4 - 14(n\text{–}1)^3 + 47(n\text{–}1)^2 + 110(n-1) - 600]$$

$$+ \frac{1}{6}[(n\text{–}2)^3 - 9(n\text{–}2)^2 + 20(n\text{–}2) + 12$$

$$+ \frac{1}{2}[(n\text{–}3)^2 - 5(n\text{–}3)^2 + 2] + n\text{–}4$$

$$= \frac{1}{24}[n^4 - 4n^3 + 6n^2 - 4n + 1 - 14n^3 + 42n^2 - 42n + 14 + 47n^2 - 94n + 47 + 110n - 110 - 600] + (n\text{–}1) - 600]$$

$$+ \frac{1}{6}[n^3 - 6n^2 + 12n - 8 - 9n^2 + 36n - 4n + 20n - 40 + 12]$$

$$+ \frac{1}{2}[n^2 - 6n + 9 - 5n + 15 + 2] + n - 4$$

$$= \frac{1}{24}[n^4 - 18n^3 + 95n^2 - 30n - 648] +$$

$$\frac{1}{6}[n^3 - 15n^2 + 68n - 72] + \frac{1}{2}[n^2 - 11n + 26] + n\text{–}4$$

$$\frac{1}{6}[n^4 - 18n^3 + 95n^2 - 30n - 648 + 4n^3 - 60n^2 + 272n - 288 + 12n^2 - 132n + 312 + 24n - 96]$$

$$= \frac{n^3 - 14n^3 + 47n^2 + 134n - 720}{24}$$

Hence the result is true for all n.

3.3 Theorem

$$\Sigma_{i=n}^{3n} d_g(C_i, n) = 3\,\Sigma_{i=4}^{3n\text{-}3} d_g(C_i, n\text{–}1)$$

Proof

$$\Sigma_{i=4}^{12} d_g(C_i, 4) = 45 = 3\,\Sigma_{i=4}^{9} d_g(C_i, 3)$$

$$\Sigma_{i=k}^{3k} d_g(C_i, k) = \Sigma_{i=k}^{3k} d_g(C_{i\text{-}1}, k-1) + \Sigma_{i=k}^{3k} d_g(C_{i\text{-}2}, k-1) + \Sigma_{i=k}^{3k} d_g(C_{i-3}, k-1)$$

$$= 3\,\Sigma_{i=k-1}^{3(k-1)} d_g(C_{i-1}, k-2) + 3\,\Sigma_{i=k-1}^{3(k-1)} d_g(C_{i-2}, k-2) +$$

$$3\,\Sigma_{i=k-1}^{3(k-1)}\ d_g(C_{i-3}, k-2)$$
$$= 3\,\Sigma_{i=k-1}^{3(k-3)}\ d_g(C_i, k-1)$$

We have the result.

IV. REFERENCES

[1] S.Alikhani, and Y.H.Peng. (2009 May 14). "Introduction to domination polynomial of a Graph." .Ar. Xiv : 0905.2251 v1[math.co].

[2] S.Alikhani, and Y.H.Peng. (2009), "Domination sets and Domination Polynomialsof Paths." *International journal of Mathematics and Mathematical Sciences.* Article ID 542040.

[3] S.Alikhani and Y.H.Peng. (20 May 2000). Domination sets and Domination Polynomials of cycles, arXiv: 0905.3268v [math.co]

[4] G.Chartand, and P.Zhang.(2005). *Introduction* to graph theory. McGraw-Hill Boston, Mass,USA.

[5] T.W.Haynes, S.T.Hedetniemi, and P.J.Slater. (1998). Fundamentals of Domination in Graphs, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker,New York,NY,Usa,1998